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Zeros of Analytic Functions with Restricted Coefficients

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Abstract— In this paper, we consider the problem of finding the number of zeros of a special class of analytic functions of polynomial functions in prescribed regions, by subjecting the certain restrictions on the real and imaginary coefficient for a certain stage by taking even and odd cases in our results. Our results generalized the earlier known results with the hypothesis about some special restriction on the real part and monotonic on imaginary parts of the polynomial coefficients and improved many theorems and corollaries. We have been working with the Eneström-Kakeya theorem hypothesis by about the number of zeros of a polynomial with restrictions. As special cases, the extended results yield much simpler expressions for the upper bounds of zeros of those existing results with the different types of restrictions on the coefficients of a polynomial and improved the many theorems and corollaries by taking analytic functions with complex coefficient.

***Keywords*:** Zeros of polynomial, Analytic functions, Eneström-Kakeya theorem, Number of Zeros, Regions

I. INTRODUCTION

The famous result is known as Eneström-Kakeya [1-2]. In this literature [3-5] there exist extensions and generalizations of the Eneström-Kakeya theorem. Finding approximate zeros of polynomial related to analytic function is an important and well-studied problem. To find the number of zeros polynomial related analytic function has already proved [6], by extending the Eneström-Kakeya theorem

Subsequently we have also generalized several results [6-9] by constructing various coefficients.

The purpose of this research paper is to generalize and extend location of zeros of analytic and number of zeros of analytic functions which are more interesting. We establish the following results.

II. RELATED WORK

Aziz and Mohammad [3] generalized the Enestrom Kakeya Theorem in a different way and proved the following by using Schwartz lemma.

Theorem 2.1. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \le t$.
If $a_i > 0$ and $a_i - t a_i \ge 0$, for $i = 1,2,3, \ldots$ then $F(z)$ does not vanish in $|z| \le t$.

Our results will be discussed by using the following results to prove our theorems.

Lemma 2.2. [4]: Let $P(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \le 1$ such that

$$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2};\ |a_{i-1}| \le |a_i| \text{ for}$$

$i = 0,1,2,\ldots,n,\ldots.$ Then
$|a_i - a_{i-1}| \le (|a_i| - |a_{i-1}|)\cos\alpha + (|a_i| + |a_{i-1}|)\sin\alpha.$

Lemma 2.3. [5]: If $f(z)$ is regular $f(0) \ne 0$ and $f(z) \le M$ in $|z| \le 1$, then the nuber of zeros of $f(z)$ in $|z| \le r$,

$$0 < r < 1 \text{ does not exceed } \frac{1}{\log\frac{1}{r}} \log\frac{M}{|a_0|}.$$

III. MAIN RESULTS AND DISCUSSION

Theorem 3.1. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \le 1$ such that

$$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}, i = 0,1,2,\ldots,n,\ldots \text{ for some real } \beta, a_0 \ne 0, k \ge 0 \text{ and}$$

$$|a_0| \ge |a_1| \le |a_2| \ge |a_3| \le |a_4| \ge \cdots \le |a_{n-2}| \ge |a_{n-1}| \le k|a_n| \ge |a_{n+1}| \ge \cdots \text{ if n is even}$$

OR

$$|a_0| \le |a_1| \ge |a_2| \le |a_3| \ge |a_4| \le \cdots \le |a_{n-2}| \ge |a_{n-1}| \le k|a_n| \ge |a_{n+1}| \ge \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \leq r,\ 0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{\begin{bmatrix}(|a_0| + 2(k-1)|a_n|)(1+\cos\alpha+\sin\alpha)\\ +2\cos\alpha\sum_{i=1}^{\frac{n}{2}}(|a_{2i}|-|a_{2i-1}|) + 2\sin\alpha\sum_{i=1}^{\infty}|a_i|\end{bmatrix}}{|a_0|}.$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \leq r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{\begin{bmatrix}(|a_0| + 2(k-1)|a_n|)(1+\cos\alpha+\sin\alpha)\\ +2\cos\alpha\sum_{i=1}^{\frac{n+1}{2}}(|a_{2i-1}|-|a_{2i-2}|)\\ +2\sin\alpha\sum_{i=1}^{\infty}|a_i|\end{bmatrix}}{|a_0|}.$$

Proof: Let $F(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + \cdots$ be analytic function.
Let us consider the polynomial $G(z) = (1-z)F(z)$ so that

$$G(z) = (1-z)(a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + \cdots)$$
$$= a_0 + \sum_{i=1}^{\infty}(a_i - a_{i-1})z^i.$$

Now for $|z| \leq 1$, we have

$$\begin{aligned}|G(z)| \leq |a_0| &+ |a_1 - a_0| + |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_{n-1} - a_{n-2}| \\ &+ |a_n - ka_n + ka_n - a_{n-1}| + |a_{n+1} - ka_n + ka_n - a_n| + \sum_{i=n+2}^{\infty}|a_i - a_{i-1}| \\ \leq |a_0| &+ |a_1 - a_0| + |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_{n-1} - a_{n-2}| + 2(k-1)|a_n| \\ &+ |ka_n - a_{n-1}| + |ka_n - a_{n+1}| + \sum_{i=n+2}^{\infty}|a_i - a_{i-1}|\end{aligned}$$

By using lemma 1 we get

$$\begin{aligned}|G(z)| \leq |a_0| &+ (|a_0|-|a_1|)\cos\alpha + (|a_0|+|a_1|)\sin\alpha + (|a_2|-|a_1|)\cos\alpha + (|a_2|+|a_1|)\sin \\ &+ (|a_2|-|a_3|)\cos\alpha + (|a_2|+|a_3|)\sin\alpha + \cdots + (|a_{n-2}|-|a_{n-1}|)\cos\alpha + (|a_{n-2}|+|a_{n-1}|)\sin\alpha \\ &+ (k|a_n|-|a_{n-1}|)\cos\alpha + (k|a_n|+|a_{n-1}|)\sin\alpha + 2(k-1)|a_n| \\ &+ (k|a_n|-|a_{n+1}|)\cos\alpha + (k|a_n|+|a_{n+1}|)\sin\alpha \\ &+ \sum_{i=n+2}^{\infty}(|a_{i-1}|-|a_i|)\cos\alpha + \sum_{i=n+2}^{\infty}(|a_i|+|a_{i-1}|)\sin\alpha,\end{aligned}$$

if n is even

$$\begin{aligned}&= (|a_0| + 2k|a_n|)(1+\cos\alpha+\sin\alpha) + 2\cos\alpha\sum_{i=1}^{\frac{n}{2}}(|a_{2i}|-|a_{2i-1}|) + 2\sin\alpha\sum_{i=1}^{\infty}|a_i| \\ &\quad - 2|a_n|(1+\sin\alpha+\cos\alpha)\end{aligned}$$

$$= \Big[(|a_0| + 2(k-1)|a_n|)(1+\cos\alpha+\sin\alpha) + 2\cos\alpha\sum_{i=1}^{\frac{n}{2}}(|a_{2i}|-|a_{2i-1}|) + 2\sin\alpha\sum_{i=1}^{\infty}|a_i|\Big].$$

Apply lemma 2 to G(z), we get then number of zeros of $G(z)$ in $|z| \leq r, 0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{\begin{bmatrix}(|a_0| + 2(k-1)|a_n|)(1+\cos\alpha+\sin\alpha)\\ +2\cos\alpha\sum_{i=1}^{\frac{n}{2}}(|a_{2i}|-|a_{2i-1}|) + 2\sin\alpha\sum_{i=1}^{\infty}|a_i|\end{bmatrix}}{|a_0|}$$

if n is even.

All the number of zeros of $F(z)$ in $|z| \leq r, 0 < r < 1$ is also equal to the number of zeros of $G(z)$ in $|z| \leq r, 0 < r < 1$, if n is even.

Similarly we can also prove for odd degree polynomials by re-arranging terms in above proof.
That is if n is odd Apply lemma 2 to G(z), we get then number of zeros of $G(z)$ in $|z| \leq r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{\begin{bmatrix}(|a_0| + 2(k-1)|a_n|)(1+\cos\alpha+\sin\alpha)\\ +2\cos\alpha\sum_{i=1}^{\frac{n+1}{2}}(|a_{2i-1}|-|a_{2i-2}|)\\ +2\sin\alpha\sum_{i=1}^{\infty}|a_i|\end{bmatrix}}{|a_0|}$$

if n is odd.

All the number of zeros of $F(z)$ in $|z| \leq r, 0 < r < 1$ is also equal to the number of zeros of $G(z)$ in $|z| \leq r, 0 < r < 1$, if n is odd.

This completes the proof of theorem 3.1.

Corollary 3.2.. Let $F(z) = \sum_{i=0}^{\infty} a_iz^i \not\equiv 0$ be analytic function in $|z| \leq 1$ such that

$$|\arg(a_i) - \beta| \leq \alpha \leq \frac{\pi}{2}, i = 0,1,2,\ldots,n,\ldots \text{ for some real } \beta, a_0 \neq 0 \text{ and}$$

$$|a_0| \geq |a_1| \leq |a_2| \geq |a_3| \leq |a_4| \geq \cdots \leq |a_{n-2}| \geq |a_{n-1}| \leq |a_n| \geq |a_{n+1}| \geq \cdots \text{ if n is even}$$

OR

$$|a_0| \le |a_1| \ge |a_2| \le |a_3| \ge |a_4| \le \cdots \le |a_{n-2}| \ge |a_{n-1}| \le |a_n| \ge |a_{n+1}| \ge \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{\left[\begin{array}{c} |a_0|(1+\cos\alpha+\sin\alpha) \\ +2\cos\alpha \sum_{i=1}^{\frac{n}{2}} (|a_{2i}| - |a_{2i-1}|) \\ +2\sin\alpha \sum_{i=1}^{\infty} |a_i| \end{array}\right]}{|a_0|}.$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{\left[\begin{array}{c} |a_0|(1+\cos\alpha+\sin\alpha) \\ +2\cos\alpha \sum_{i=1}^{\frac{n+1}{2}} (|a_{2i-1}| - |a_{2i-2}|) \\ +2\sin\alpha \sum_{i=1}^{\infty} |a_i| \end{array}\right]}{|a_0|}$$

Corollary 3.3. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \le 1$ such that

$$a_0 \ne 0, k \ge 0 \text{ and}$$

$$|a_0| \ge |a_1| \le |a_2| \ge |a_3| \le |a_4| \ge \cdots \le |a_{n-2}| \ge |a_{n-1}| \le k|a_n| \ge |a_{n+1}| \ge \cdots \text{ if n is even}$$

OR

$$|a_0| \le |a_1| \ge |a_2| \le |a_3| \ge |a_4| \le \cdots \le |a_{n-2}| \ge |a_{n-1}| \le k|a_n| \ge |a_{n+1}| \ge \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{\left[\begin{array}{c} (|a_0| + 2(k-1)|a_n|) \\ + \sum_{i=1}^{\frac{n}{2}} (|a_{2i}| - |a_{2i-1}|) \end{array}\right]}{|a_0|}.$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{\left[\begin{array}{c} (|a_0| + 2(k-1)|a_n|) \\ + \sum_{i=1}^{\frac{n+1}{2}} (|a_{2i-1}| - |a_{2i-2}|) \end{array}\right]}{|a_0|}$$

Remark 3.4. By taking $k = 1$ and $r = \frac{1}{2}$ in theorem 3.1, then it reduces to Corollary 3.2.

Remark 3.5. By taking $\alpha = \beta = 0$ and $r = \frac{1}{2}$ in theorem 3.1, then it reduces to Corollary 3.3.

Theorem 3.6. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \le 1$ such that

$$a_i = \alpha_i + i\beta_i, \beta_i \ge \beta_{i+1} \text{ for } i = 0,1,2\ldots, a_0 \ne 0, k \ge 0 \text{, and}$$

$$\alpha_0 \ge \alpha_1 \le \alpha_2 \ge \alpha_3 \le \alpha_4 \ge \cdots \le \alpha_{n-2} \ge \alpha_{n-1} \le k\alpha_n \ge \alpha_{n+1} \ge \cdots \text{ if n is even}$$

OR

$$\alpha_0 \le \alpha_1 \ge \alpha_2 \le \alpha_3 \ge \alpha_4 \le \cdots \le \alpha_{n-2} \ge \alpha_{n-1} \le k\alpha_n \ge \alpha_{n+1} \ge \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \le r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log \frac{1}{r}} \log \left[\frac{\begin{array}{c} |\alpha_0| + |\beta_0| - \alpha_0 + \beta_0 \\ +2[\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-2} - \alpha_{2i-1}) + k(\alpha_n + |\alpha_n|) - |\alpha_n|] \end{array}}{|a_0|}\right].$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \le r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log \frac{1}{r}} \log \left[\frac{\begin{array}{c} |\alpha_0| + |\beta_0| - \alpha_0 + \beta_0 \\ +2[\sum_{i=1}^{\frac{n-1}{2}} (\alpha_{2i-1} - \alpha_{2i}) \\ +k(\alpha_n + |\alpha_n|) - |\alpha_n|] \end{array}}{|a_0|}\right].$$

Proof: Let $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + a_{n+1} z^{n+1} + \cdots$ be analytic function

Let us consider the polynomial $G(z) = (z-1)F(z)$ so that

$$G(z) = (z-1)(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + a_{n+1} z^{n+1} + \cdots)$$

$$= -a_0 + \sum_{i=1}^{\infty} (a_{i-1} - a_i) z^i.$$

Now for $|z| \le 1$, we have

$$|G(z)| \le |a_0| + \sum_{i=1}^{\infty} |a_{i-1} - a_i|$$

$$\le |\alpha_0| + |\beta_0| + \sum_{i=1}^{\infty} |\alpha_{i-1} - \alpha_i| + \sum_{i=1}^{\infty} |\beta_{i-1} - \beta_i|$$

$$= |\alpha_0| + |\beta_0| + |\alpha_0 - \alpha_1| + |\alpha_1 - \alpha_2| + |\alpha_2 - \alpha_3| + |\alpha_3 - \alpha_4| + \cdots + |\alpha_{n-2} - \alpha_{n-1}| + |\alpha_{n-1} - k\alpha_n + k\alpha_n - \alpha_n| + |\alpha_n - k\alpha_n + k\alpha_n - \alpha_{n+1}| + \sum_{i=n+2}^{\infty} |\alpha_{i-1} - \alpha_i| + \sum_{i=1}^{\infty} |\beta_{i-1} - \beta_i|$$

$$\leq |\alpha_0| + |\beta_0| + (\alpha_0 - \alpha_1) + (\alpha_2 - \alpha_1) + (\alpha_2 - \alpha_3) + (\alpha_4 - \alpha_3) + \cdots + (k\alpha_n - \alpha_{n-1}) + 2(k-1)|\alpha_n| + (k\alpha_n - \alpha_{n+1}) + \sum_{i=n+2}^{\infty} (\alpha_{i-1} - \alpha_i) + \sum_{i=2}^{\infty} (\beta_{i-1} - \beta_i)$$

$$\leq |\alpha_0| + |\beta_0| - \alpha_0 + \beta_0 + 2[\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-2} - \alpha_{2i-1}) + k(\alpha_n + |\alpha_n|) - |\alpha_n|]$$

Apply lemma 2 to G(z), we get then number of zeros of $G(z)$ in $|z| \leq r, 0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \left[\frac{|\alpha_0| + |\beta_0| - \alpha_0 + \beta_0 + 2[\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-2} - \alpha_{2i-1}) + k(\alpha_n + |\alpha_n|) - |\alpha_n|]}{|a_0|} \right],$$

if n is even.

All the number of zeros of $F(z)$in $|z| \leq r$, $0 < r < 1$ is also equal to the number of zeros of $G(z)$ in $|z| \leq r$, $0 < r < 1$, if n is even.

Similarly we can also prove for odd degree polynomials by re-arranging terms in above proof.

That is if n is odd Apply lemma 2 to G(z), we get then number of zeros of $G(z)$ in $|z| \leq r, 0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \left[\frac{|\alpha_0| + |\beta_0| - \alpha_0 + \beta_0 + 2[\sum_{i=1}^{\frac{n-1}{2}} (\alpha_{2i-1} - \alpha_{2i}) + k(\alpha_n + |\alpha_n|) - |\alpha_n|]}{|a_0|} \right],$$

if n is odd.

All the number of zeros of $F(z)$ in $|z| \leq r, 0 < r < 1$ is also equal to the number of zeros of $G(z)$ in $|z| \leq r$, $0 < r < 1$, if n is odd.

This completes the proof of theorem 3.6.

Corollary 3.7. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \leq 1$ such that

$$a_i = \alpha_i + i\beta_i, \beta_i \geq \beta_{i+1} \text{ for } i = 0,1,2 \ldots, a_0 \neq 0 \text{ and}$$
$$\alpha_0 \geq \alpha_1 \leq \alpha_2 \geq \alpha_3 \leq \alpha_4 \geq \cdots \leq \alpha_{n-2} \geq \alpha_{n-1} \leq \alpha_n \geq \alpha_{n+1} \geq \cdots \text{ if n is even}$$

OR

$$\alpha_0 \leq \alpha_1 \geq \alpha_2 \leq \alpha_3 \geq \alpha_4 \leq \cdots \leq \alpha_{n-2} \geq \alpha_{n-1} \leq \alpha_n \geq \alpha_{n+1} \geq \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \left[\frac{|\alpha_0| + |\beta_0| - \alpha_0 + \beta_0 + 2[\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i} - \alpha_{2i-1})]}{|a_0|} \right].$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \left[\frac{|\alpha_0| + |\beta_0| - \alpha_0 + \beta_0 + 2[\sum_{i=0}^{\frac{n-1}{2}} (\alpha_{2i+1} - \alpha_{2i})]}{|a_0|} \right].$$

Remark 3.8. By taking $k = 1$ and $r = \frac{1}{2}$ in theorem 3.6, then it reduces to Corollary 3.7.

Theorem 3.9. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \leq 1$ such that

$$a_i = \alpha_i + i\beta_i, \text{for } i = 0,1,2 \ldots, a_0 \neq 0, 0 < \tau \leq 1, \text{and}$$
$$\alpha_0 \leq \alpha_1 \geq \alpha_2 \leq \alpha_3 \geq \alpha_4 \leq \cdots \geq \alpha_{n-2} \leq \alpha_{n-1} \geq \tau\alpha_n \leq \alpha_{n+1} \geq \alpha_{n+2} \geq \cdots \text{ if n is even}$$

OR

$$\alpha_0 \geq \alpha_1 \leq \alpha_2 \geq \alpha_3 \leq \alpha_4 \geq \cdots \geq \alpha_{n-2} \leq \alpha_{n-1} \geq \tau\alpha_n \leq \alpha_{n+1} \geq \alpha_{n+2} \geq \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \leq r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \left[\frac{|\alpha_0| + \alpha_0 + 2[\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \alpha_{2i-2}) + \alpha_{n+1} - \tau(\alpha_n + |\alpha_n|) + |\alpha_n| + \sum_{i=0}^{\infty} |\beta_i|]}{|a_0|} \right].$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \leq r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \left[\frac{|\alpha_0| - \alpha_0 + 2[\sum_{i=0}^{\frac{n-1}{2}} (\alpha_{2i} - \alpha_{2i+1}) + \alpha_{n+1} - \tau(\alpha_n + |\alpha_n|) + |\alpha_n| + \sum_{i=0}^{\infty} |\beta_i|]}{|a_0|} \right].$$

Proof: Let $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + a_{n+1} z^{n+1} + \cdots$ be analytic function

Let us consider the polynomial $G(z) = (z-1)F(z)$ so that

$$G(z) = (z-1)(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + a_{n+1} z^{n+1} + \cdots)$$
$$= -a_0 + \sum_{i=1}^{\infty} (a_{i-1} - a_i) z^i.$$

Now for $|z| \le 1$, we have

$$|G(z)| \le |a_0| + \sum_{i=1}^{\infty} |a_{i-1} - a_i|$$

$$\le |\alpha_0| + |\beta_0| + \sum_{i=1}^{\infty} |\alpha_{i-1} - \alpha_i| + \sum_{i=1}^{\infty} |\beta_{i-1} - \beta_i|$$

$$\le |\alpha_0| + |\beta_0| + |\alpha_0 - \alpha_1| + |\alpha_1 - \alpha_2| + |\alpha_2 - \alpha_3| + |\alpha_3 - \alpha_4| + \cdots + |\alpha_{n-2} - \alpha_{n-1}| + |\alpha_{n-1} - \tau\alpha_n + \tau\alpha_n - \alpha_n| + |\alpha_n - \tau\alpha_n + \tau\alpha_n - \alpha_{n+1}| + \sum_{i=n+2}^{\infty} |\alpha_{i-1} - \alpha_i| + \sum_{i=1}^{\infty} (|\beta_{i-1}| + |\beta_i|)$$

$$\le |\alpha_0| + (\alpha_1 - \alpha_0) + (\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_2) + (\alpha_3 - \alpha_4) + \cdots + (\alpha_{n-1} - \tau\alpha_n) + 2(1-\tau)|\alpha_n| + (\alpha_{n+1} - \tau\alpha_n) + \sum_{i=n+2}^{\infty} (\alpha_{i-1} - \alpha_i) + 2\sum_{i=0}^{\infty} |\beta_i|$$

$$\le |\alpha_0| + \alpha_0 + 2[\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \alpha_{2i-2}) + \alpha_{n+1} - \tau(\alpha_n + |\alpha_n|) + |\alpha_n| + \sum_{i=0}^{\infty} |\beta_i|]$$

Apply lemma 2 to G(z), we get then number of zeros of $G(z)$ in $|z| \le r, 0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log\left[\frac{|\alpha_0| + \alpha_0 + 2[\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \alpha_{2i-2}) + \alpha_{n+1} - \tau(\alpha_n + |\alpha_n|) + |\alpha_n| + \sum_{i=0}^{\infty} |\beta_i|]}{|a_0|}\right],$$

if n is even.

All the number of zeros of $F(z)$ in $|z| \le r, 0 < r < 1$ is also equal to the number of zeros of $G(z)$ in $|z| \le r$, $0 < r < 1$, if n is even

Similarly we can also prove for odd degree polynomials by re-arranging terms in above proof.
That is if n is odd Apply lemma 2 to G(z), we get then number of zeros of $G(z)$ in $|z| \le r, 0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log\left[\frac{|\alpha_0| - \alpha_0 + 2[\sum_{i=0}^{\frac{n-1}{2}} (\alpha_{2i} - \alpha_{2i-1}) + \alpha_{n+1} - \tau(\alpha_n + |\alpha_n|) + |\alpha_n| + \sum_{i=0}^{\infty} |\beta_i|]}{|a_0|}\right],$$

if n is odd.

All the number of zeros of $F(z)$ in $|z| \le r, 0 < r < 1$ is also equal to the number of zeros of $G(z)$ in $|z| \le r$, $0 < r < 1$, if n is odd.

This completes the proof of theorem 3.9.

Corollary 3.10. . Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \le 1$ such that

$$a_i = \alpha_i + i\beta_i, \text{for } i = 0,1,2 \ldots, a_0 \ne 0, \text{and}$$
$$\alpha_0 \le \alpha_1 \ge \alpha_2 \le \alpha_3 \ge \alpha_4 \le \cdots \ge \alpha_{n-2} \le \alpha_{n-1} \ge \alpha_n \le \alpha_{n+1} \ge \alpha_{n+2} \ge \cdots \text{ if n is even}$$
OR
$$\alpha_0 \ge \alpha_1 \le \alpha_2 \ge \alpha_3 \le \alpha_4 \ge \cdots \ge \alpha_{n-2} \le \alpha_{n-1} \ge \alpha_n \le \alpha_{n+1} \ge \alpha_{n+2} \ge \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log\left[\frac{|\alpha_0| + \alpha_0 + 2[\sum_{i=0}^{\frac{n}{2}} (\alpha_{2i+1} - \alpha_{2i}) + \sum_{i=0}^{\infty} |\beta_i|]}{|a_0|}\right].$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log\left[\frac{|\alpha_0| - \alpha_0 + 2[\sum_{i=1}^{\frac{n+1}{2}} (\alpha_{2i} - \alpha_{2i-1}) + \sum_{i=0}^{\infty} |\beta_i|]}{|a_0|}\right].$$

Remark 3.11. By taking $\tau = 1$ and $r = \frac{1}{2}$ in theorem 3.9, then it reduces to Corollary 3.10.

By re-arrangement of terms in above two theorems we get following theorem.

Theorem 3.12. . Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ be analytic function in $|z| \le 1$ such that

$$a_i = \alpha_i + i\beta_i, \text{for } i = 0,1,2 \ldots, a_0 \ne 0 \text{ and}$$
$$\alpha_0 \ge \alpha_1 \le \alpha_2 \ge \alpha_3 \le \alpha_4 \ge \cdots \le \alpha_{n-2} \ge \alpha_{n-1} \le \alpha_n \ge \alpha_{n+1} \ge \cdots \text{ if n is even}$$

OR

$$\alpha_0 \le \alpha_1 \ge \alpha_2 \le \alpha_3 \ge \alpha_4 \le \cdots \le \alpha_{n-2} \ge \alpha_{n-1} \le \alpha_n \ge \alpha_{n+1} \ge \cdots \text{ if n is odd}$$

then (i) If n is even the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left[\frac{|\alpha_0| + \alpha_0 + 2[\sum_{i=1}^{\frac{n}{2}}(\alpha_{2i} - \alpha_{2i-1}) + \sum_{i=0}^{\infty}|\beta_i|]}{|a_0|}\right].$$

(ii) If n is odd the number of zeros of $F(z)$ in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left[\frac{|\alpha_0| + \alpha_0 + 2[\sum_{i=0}^{\frac{n-1}{2}}(\alpha_{2i+1} - \alpha_{2i}) + \sum_{i=0}^{\infty}|\beta_i|]}{|a_0|}\right].$$

Proof: Proof of this theorem is similar to the proof of above theorems.

IV. CONCLUSION AND FUTURE SCOPE OF WORK

In this research paper we, generalized various known results and established the number zeros of analytic function with restricted coefficients in various cases by constructing even and odd cases.

If someone interested to do in this paper they may consider different types of coefficients and set up the different results.

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REFERENCES

[1]. G.Eneström, Remarquee sur un théorème relatif aux racines de l'equation a_n+...+a_0=0 oùtous les coefficient sont et positifs, Tôhoku Math.J 18(1920),34-36.

[2]. S.KAKEYA, On the limits of the roots of an alegebraic equation with positive coefficient, Tôhoku Math.J 2 (1912-1913),140-142.

[3]. A. Aziz and Q.G. Mahammad, on the certain class of analytic class of olynomial and related analytic functions, J. Anal and Appl. 75,[1980], 495-502.

[4]. N.K.Govil and Q.I.Rehman, On the EnstromKakeya Theorem Tohoku Math .J. 20, (1968), 126-136.

[5]. K. K. Dewan, Extremal Properties and Coefficient Estimates for Polynomials with Restricted Zeros and on Location of Zeros of Polynomials, Ph.D Thesis, Indian Institutes of Technology, Delhi, 1980.

[6]. P.Ramulu, G.L.Reddy, On the zeros of polynomials, Int Jr. of Mathematical Sciences & Applications, Vol. 5, No. 2, (July-December, 2015), pp-395-399.

[7]. P. Ramulu, Some Generalization of Eneström-Kakeya Theorem: International Journal of Mathematics and Statistics Invention IJMSI), (2015), 3 (2), 52-59.

[8]. P. Ramulu, G. L. Reddy and C.Gangadhar on the zeros of polynomials with real coefficients: International Journal of Applied Mathematical Sciences. (2015), 5 (2), 109-118.

[9]. G.L.Reddy, and P. Ramulu On the zeros of Polynomials: International Journal Mathematical Sciences and Applications, (2015), 5 (2), 395-399.

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